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Internal waves in a viscous atmosphere

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The way in which internal waves change in amplitude as they propagate through an incompressible fluid or an isothermal atmosphere is considered. A similarity solution for the small amplitude isolated viscous internal wave which is generated by a localized two-dimensional disturbance or energy source was given by Thomas & Stevenson (1972). It will be shown how summations or superpositions of this solution may be used to examine the behaviour of groups of internal waves. In particular the paper considers the waves produced by an infinite number of sources distributed in a horizontal plane such that they produce a sinusoidal velocity distribution. The results of this analysis lead to a new small perturbation solution of the linearized equations.

1. Introduction

The propagation of small amplitude waves in a non-dissipative stably stratified fluid has been studied by Moore & Spiegel (1964), Midgley & Liemohn (1966) and many others (see the review by Yeh & Liu 1974). The system of linearized equations governing the motion of small amplitude waves can be reduced to ordinary differential equations by considering a wave form which is sinusoidal in the horizontal co-ordinate. One difficulty with the inviscid solutions is that the amplitudes of the internal waves increase exponentially with height.

The effects of an exponentially increasing kinematic viscosity on the damping and reflexion of waves in an incompressible atmosphere was studied by Yanowitch (1967). He showed that, as the altitude increases and the viscosity becomes important, the wave amplitudes increase to a maximum and then reduce to zero at high altitudes. At the high altitudes where viscosity is important, energy is reflected back towards the ground. Yanowitch's solution at low altitudes is essentially an inviscid solution in which the amplitude ratio of the reflected wave and the wave with upgoing energy is given by $\exp(-2\pi^2 H/\lambda_z)$, where λ_z is the wavelength in the vertical direction and H is the stratification scale height. Consequently the reflected waves become important only when the wavelength is large compared with the scale height.

For a viscous heat-conducting isothermal atmosphere the altitude at which the maximum amplitude occurs was obtained by a (not strictly valid) perturbation analysis by Pitteway & Hines (1963) and numerically by Midgley & Liemohn (1966). However the variation in the amplitude with altitude was not given. Yanowitch's theory has been extended to an isothermal atmosphere by Lyons & Yanowitch (1974) for very small Prandtl numbers. Warren (1972) has considered wave reflexions in an isothermal atmosphere and his results agree qualitatively with Yanowitch's incompressible theory. Warren's amplitude ratio of reflected and incident waves for an isothermal atmosphere is $\exp(-2\pi^2 H/3\lambda_z)$. Lindzen (1970, 1971) and Lindzen & Blake (1971) included the effects of radiation in an atmosphere with arbitrary distributions of background temperature. Their numerical solutions for an isothermal atmosphere show that the wave amplitude increases with altitude before eventually tending to a constant value.

Similarity solutions for an isolated internal wave were obtained for an incompressible viscous fluid by Thomas & Stevenson (1972, 1973) and for a viscous heat-conducting isothermal atmosphere by Stevenson, Bearon & Thomas (1974). These are solutions for the wave which develops about a localized oscillatory disturbance. It is these solutions which are used in the present paper and it will be shown that, by distributing an infinite number of the 'isolated' waves, the problem studied by Pitteway & Hines (1963) can be solved. The solutions have λ_z no larger than twice the stratification scale height and it is found that the wave amplitudes approach zero at large altitudes without the addition of any reflected waves. This is compatible with Yanowitch's and Warren's analyses, which show that wave reflexions are negligible for these wavelengths.

First of all an incompressible fluid will be considered and a superposition of the isolated waves will be evaluated numerically. An approximate analytical integration of the isolated waves is then presented and it is this solution which gives the clue to a small perturbation analysis of the linearized incompressible equations. Solutions for an isothermal atmosphere are then approached in a similar manner.

2. Incompressible fluid

2.1. *A summation of the isolated waves*

Thomas & Stevenson (1972) considered an incompressible stratified fluid with a constant buoyancy frequency ω_0. A horizontal cylinder oscillating at a frequency ω which is less than ω_0 produces an internal wave which looks like a St Andrew's cross. A similarity solution was obtained which showed that the amplitudes along the cross vary as $X_1^{-\frac{2}{3}}$, where

$$X_1 = \int_0^{X'} \nu_1(\bar{X}')\,d\bar{X}'. \tag{1}$$

X' is the dimensionless distance defined by $X' = X\beta\sin\theta$, where X is the true distance, with the X axis inclined at an angle θ to the horizontal (figure 1). ν_1 is the value of the undisturbed kinematic viscosity on the X axis divided by the kinematic viscosity ν^* at the virtual origin of the wave, where $X = 0$. $\beta = \omega_0^2/g$ is the inverse scale height and g is the gravitational acceleration. The background density distribution ρ_0 is given by $\rho_0 = \rho^*\exp(-\beta z)$ with the z axis vertical (figure 1). The undisturbed density distribution along the X axis is given by

$$r_1 = \rho(X')/\rho^* = \exp(-X'). \tag{2}$$

The starred conditions are constant reference conditions at $X = 0$. The velocity component along the wave is given by

$$U = \left(\frac{ag}{\omega_0}\right) r_1^{-\frac{1}{2}} X_1^{-\frac{2}{3}}\,\mathrm{Re}\left\{\exp(-i\omega t)\int_0^\infty K\exp(iK\eta - K^3)\,dK\right\}, \tag{3}$$

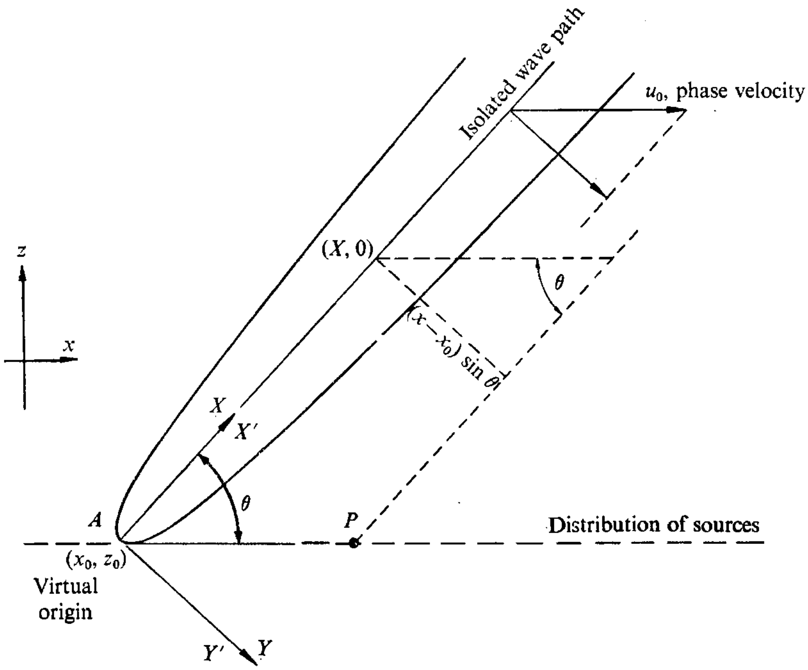

FIGURE 1. The co-ordinate systems.

where a is a constant amplitude coefficient, t is the time and η is the similarity variable:

$$\eta = (Y\beta \sin\theta/\alpha X_1^{\frac{1}{3}}). \tag{4}$$

α is a viscosity coefficient given by

$$\alpha = \{(\omega_0^3 \nu^*/2g^2)\tan\theta\sin\theta\}^{\frac{1}{3}} \tag{5}$$

and $\theta = \sin^{-1}(\omega/\omega_0)$. We consider a kinematic viscosity which varies exponentially with height, so that $\nu_1 = \exp X'$. This implies that the viscosity μ has a constant value throughout the fluid. Thus from (1)

$$X_1 = \{\exp X' - 1\}. \tag{6}$$

The velocity component V, which is in the Y direction, is found to be very small and it will be neglected in the numerical summations.

Many isolated waves having the same frequency are now placed side by side with all their sources or virtual origins in the same horizontal plane. The sources extend from $-\infty$ to $+\infty$ and their phases are consistent with a sinusoidal velocity distribution of wavelength λ in the horizontal direction. Referring to figure 1, we look for the contribution to the velocity at the point $(X, 0)$ from a source of strength $s\,\delta\bar{x}$ at P, which is at a distance $\bar{x}$ from A. A, at which $x = x_0$ and $z = z_0$, is the origin of the (X, Y) co-ordinate system. If the phase at P is given by $\exp\{i[k(x_0+\bar{x})-\omega t]\}$, where $k = 2\pi/\lambda$, then the contribution from the sources between $x = -b$ and $+b$ may be written, from (3), as

$$U = s\int_{-b}^{+b} r_1^{-\frac{1}{2}} X_1^{-\frac{2}{3}} \exp\{i[k(x_0+\bar{x})-\omega t]\}\int_0^\infty K\exp\left\{-K^3 - i\left[\frac{KY\beta\sin\theta}{\alpha X_1^{\frac{1}{3}}}\right]\right\} dK\,d\bar{x}, \tag{7}$$

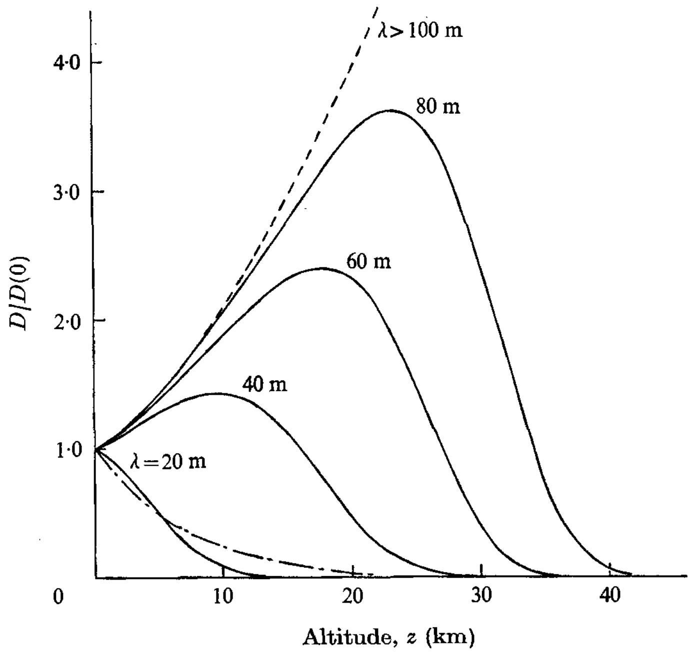

FIGURE 2. The variation of the vertical displacement amplitude D with height for an incompressible fluid. $D(0)$ is the displacement amplitude at $z = 0$. $\omega_0 = 3{\cdot}88 \times 10^{-2}$ rad/s, $\beta^{-1} = 6{\cdot}56$ km, $u_0 k/\omega_0 = 2^{-\frac{1}{2}}$ and ν^* at $z = 0$ is $6{\cdot}24 \times 10^{-5}$ m²/s. The virtual origin of the waves was at $z_0 = -1$ km. (Actually the virtual origin of the individual waves has no effect on the results as expected.) ——, computed from superposition of individual waves using (7); – – – – –, computed from superposition of individual waves using (7), also the inviscid solution; —·—·—·, from the superposition of waves using the Boussinesq approximation, coincident with the Boussinesq exponentially decreasing solution.

with

$$Y = \bar{x}\sin\theta, \quad X_1 = [\exp\{\beta\sin\theta(X-\bar{x}\cos\theta)\}-1], \quad r_1 = (X_1+1)^{-1}. \quad (8a\text{–}c)$$

Equation (7) is evaluated numerically, the first integral being truncated when further increases in $|b|$ have no noticeable effect on the velocity U to three significant figures. The equation for the amplitude of the particle displacements is the same as that for the velocity except that the phase differs by π and the constant amplitude coefficient has a different value.

In figure 2 the variation of amplitude with height is presented for several wavelengths and a particular background stratification. From the computations it was found that lines of constant phase are almost straight lines at an angle θ to the horizontal. The horizontal phase velocity u_0 of the resulting wave system is directed towards higher x (see figure 1) and $u_0 = \lambda\omega_0\sin\theta/2\pi$. Consequently, by superposition of a velocity $-u_0$ on the co-ordinate system, the flow model is equivalent to a stationary wave system through which there is a mean horizontal flow of velocity $-u_0$.

It is possible to integrate (7) approximately without resorting to computation. The width of the individual waves is very small and the integration with respect to $\bar{x}$ has contributions only from sources within a very small range of $\bar{x}$. Thus $\bar{x} \ll X$, so that the expression (8*b*) for X_1 is approximately

$$X_1 = \{\exp(X\beta \sin\theta) - 1\}. \tag{9}$$

The integral with respect to $\bar{x}$ in (7) can now be taken from $-\infty$ to $+\infty$ and, after changing the order of integration,

$$U = s r_1^{-\frac{1}{2}} X_1^{-\frac{2}{3}} \exp\{i(kx_0 - \omega t)\} \int_0^\infty K \exp(-K^3) \times \int_{-\infty}^{\infty} \exp\left[-i\left(\frac{K\beta \sin^2\theta}{\alpha X_1^{\frac{1}{3}}} - k\right)\bar{x}\right] d\bar{x}\, dK, \tag{10}$$

or
$$U = 2\pi s r_1^{-\frac{1}{2}} X_1^{-\frac{2}{3}} \exp\{i(kx_0 - \omega t)\} \int_0^\infty K \exp(-K^3)\, \delta\left[\frac{K\beta \sin^2\theta}{\alpha X_1^{\frac{1}{3}}} - k\right] dK, \tag{11}$$

where $\delta[\quad]$ is the Dirac delta function. This readily integrates to give

$$U = \frac{2\pi s r_1^{-\frac{1}{2}} \alpha^2 k}{\beta^2 \sin^4\theta} \exp\left\{\frac{-\alpha^3 k^3 X_1}{\beta^3 \sin^6\theta} + i(kx_0 - \omega t)\right\}. \tag{12}$$

The expressions for r_1, α and X_1, (2), (5) and (9) respectively, are substituted into (12) to give

$$U = U(0) \exp\left\{\frac{X\beta \sin\theta}{2} - \frac{\nu^* k^3 (\exp[X\beta \sin\theta] - 1)}{2\omega_0 \beta \sin^4\theta} + i(kx_0 - wt)\right\}. \tag{13}$$

If the waves are stationary relative to the co-ordinate system and U is written in terms of the vertical co-ordinate z with $z = 0$ at the level of the virtual origins of the individual waves then

$$U(x,z) = U(0) \exp\left\{ikx + \left(\frac{\beta}{2} - ik\cot\theta\right) z - \frac{\nu^* k^3 (\exp\beta z - 1)}{2\omega_0 \beta \sin^4\theta \cos\theta}\right\}, \tag{14}$$

where $\theta = \sin^{-1}(u_0 k/\omega_0)$ and $U(0)$ is constant.

If (13) or (14) is used to calculate curves of amplitude against altitude these curves are indistinguishable from the numerical results shown in figure 2. Curves for larger wavelengths are shown in figure 3. For large wavelengths, the amplitude initially increases exponentially with height, following the inviscid theory, before eventually attenuating. For small wavelengths, the amplitude decreases exponentially with height, a result which can be readily deduced from the Boussinesq equations. If the Boussinesq approximation had been made initially then this would have been equivalent to writing $r_1 = 1$ and $X' = X_1$ in the above equations. The solution for the vertical velocity perturbation would then be

$$U \sin\theta \simeq w(x,z) = w(0) \exp\left\{ikx - \left(ik\cot\theta + \frac{\nu^* k^3}{2\omega_0 \sin^4\theta \cos\theta}\right) z\right\}.$$

This equation can be derived directly from the Boussinesq form of the linearized Navier–Stokes equations without using the isolated-wave theory. However the details are not included because a more general solution is presented in the next section.

In this paper the waves are assumed to be excited by some mechanism at $z = 0$ or below, and there will be no attempt to match the solutions to a boundary-layer region over a ground.

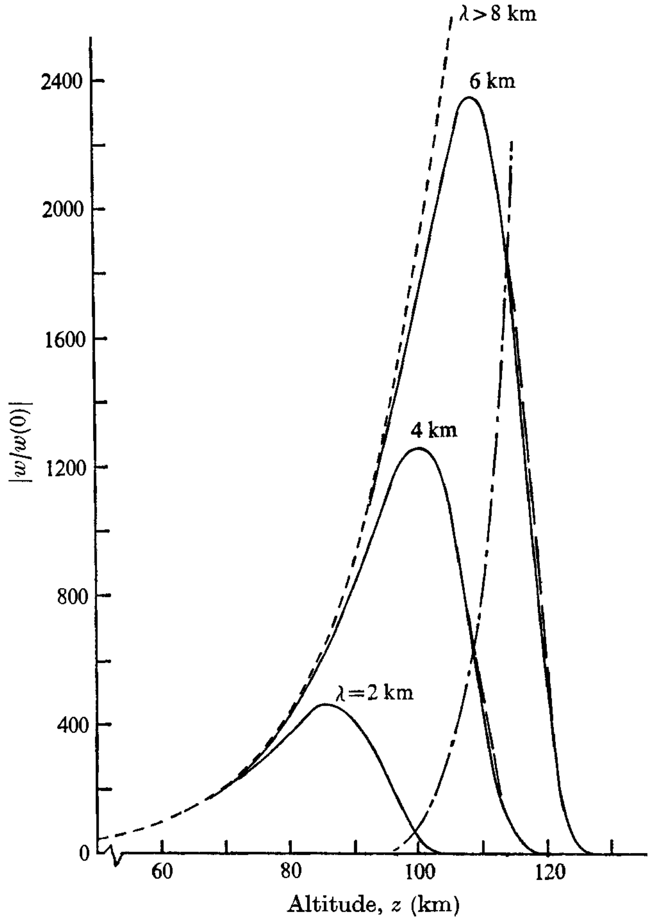

FIGURE 3. The variation of the velocity perturbation $|w(z)|$ with altitude for an incompressible fluid. The background stratification is the same as that in figure 2. ——, – – – – – –, from equation (13); — —, equation (20) using the coefficients A, B and C. The perturbation parameter $\epsilon\, e^{z_1} > 0{\cdot}1$ to the right of the chain-dashed line.

2.2. *Perturbation analysis*

For a steady flow the linearized incompressibility equation and the continuity equation are

$$\partial u/\partial x + \partial w/\partial z = 0 \tag{15}$$

and

$$-u_0\, \partial \rho/\partial x + w\, \partial \rho_0/\partial z = 0, \tag{16}$$

where $(-u_0, 0)$ is a constant mean flow velocity and u and w are the velocity perturbations in the x and z directions, with the x axis horizontal and the z axis vertically upwards. ρ is the density perturbation. The linearized momentum equations, obtained by subtracting the hydrostatic relations, are

$$\rho_0 u_0 \frac{\partial u}{\partial x} - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2}\right) = 0 \tag{17}$$

and

$$\rho_0 u_0 \frac{\partial w}{\partial x} - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial x^2}\right) - \rho g = 0. \tag{18}$$

The viscosity μ is assumed to be constant. We shall look for a wave solution which is sinusoidal in x, so that $w = w(z)\exp ikx$, $\rho = \rho(z)\exp ikx$, etc. The pressure perturbation p is eliminated between (17) and (18), ρ is eliminated using (16) and finally u is eliminated using (15). The resulting differential equation for w is

$$\mu\left\{\frac{\partial^4 w}{\partial z^4} - 2k^2\frac{\partial^2 w}{\partial z^2} + k^4 w\right\} + iku_0\left\{\frac{\partial}{\partial z}\left(\rho_0\frac{\partial w}{\partial z}\right) + \rho_0 w\left(\frac{\omega_0^2}{u_0^2} - k^2\right)\right\} = 0, \tag{19}$$

with $\rho_0 = \rho^* e^{-\beta z}$. The boundary conditions are that w and all its derivatives must approach zero as $z \to \infty$ and $w = w(0)$ at $z = 0$. In view of the results of the superposition of the individual waves, we look for a solution of the form

$$w(z_1) = w(0)\exp\{Az_1 - B[\epsilon(e^{z_1} - 1) + C\epsilon^2(e^{2z_1} - 1) + D\epsilon^3(e^{3z_1} - 1)\ldots]\}, \tag{20}$$

where

$$\epsilon = \nu^* k/u_0, \quad z_1 = z\beta. \tag{21}$$

On substituting this form for w into (19) and equating terms of like order in ϵe^{z_1} it is found that

$$A = \tfrac{1}{2} - i|\{K^2(\omega_0^2/u_0^2k^2 - 1) - \tfrac{1}{4}\}^{\frac{1}{2}}|, \tag{22}$$

$$B = -i(K^2 - A^2)^2/(2AK^2) \tag{23}$$

and

$$C = \{i[(1 + 4A + 6A^2 + 4A^3) - 2K^2(1 + 2A)] + BK^2\}/[2K^2(1 + 2A)], \tag{24}$$

where $K = k/\beta$. We shall restrict our attention to internal-wave solutions, and for these $K^2(\omega_0^2/u_0^2k^2 - 1) > \frac{1}{4}$. The sign of the square root in the expression for A corresponds to an inviscid solution without reflected waves. This will be discussed later.

When the amplitude variation with height is evaluated from (20) using the coefficients A, B and C, it is again found that there is no distinguishable difference between the results and those shown in figure 2. On further examination it is seen that, if A and B are expanded in terms of K^{-1} and if θ is introduced as $\sin^{-1}(u_0k/w_0)$, then

$$A = \left(\tfrac{1}{2} - \frac{ik\cot\theta}{\beta}\left\{1 + O\left(\frac{1}{K}\right)\right\}\right) \tag{25}$$

and

$$B = \frac{K}{2\sin^3\theta\cos\theta}\left\{1 + O\left(\frac{1}{K}\right)\right\}. \tag{26}$$

These leading terms are identical with those in (14), which was obtained from the integration of the isolated waves.

Equation (20), with the coefficients A, B and C, has been used to evaluate the amplitude–altitude curves which are plotted in figure 3. The region in which $\epsilon e^{z_1} > 0{\cdot}1$ is also shown and it is in this region that (13) and (20) differ slightly.

Equation (20) for $w(z)$ may be used to calculate mountain lee waves using an equation for the vertical velocity perturbations or for the vertical displacements of the form

$$\int_0^{k_m} f(k)\, w(z)\, dk,$$

where k_m is an upper limit on the wavenumber determined by the condition $\epsilon e^{z_1} \ll 1$. u_0 is constant and $f(k)$ is chosen to satisfy a particular ground profile.

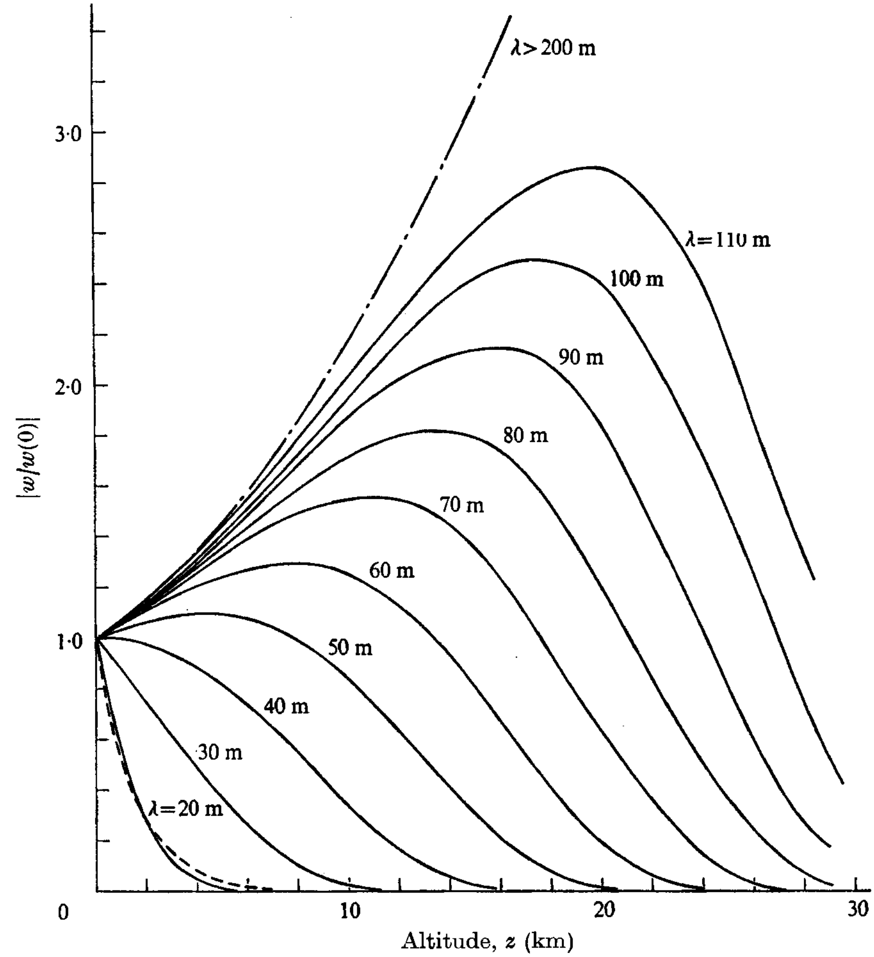

FIGURE 4. The variation of the perturbation velocity $|w(z)|$ with height for an isothermal atmosphere. $\beta = 4{\cdot}36 \times 10^{-5}\ \mathrm{m}^{-1}$, $\omega_0 = 2{\cdot}07 \times 10^{-2}$ rad/s, $\gamma = 1{\cdot}40$, $g = 9{\cdot}81\ \mathrm{m/s^2}$, $a_0 = 295$ m/s, $\sigma^* = 0{\cdot}72$, $u_0 k/\omega_0 = 2^{-\frac{1}{2}}$ and ν^* at $z = 0$ is $6{\cdot}24 \times 10^{-5}\ \mathrm{m^2/s}$. The $z = 0$ conditions correspond to the International Standard Atmosphere (ISA) at 14 km. ——, superposition of individual waves; —·—·—, superposition of individual waves, also the inviscid solution; ------, the Boussinesq solution with $r_1 = 1$ and $X_1 = X'$.

The altitude is limited by the condition $\epsilon e^{z_1} \ll 1$. C is of order unity in the calculations of the present paper but its value will depend on k and u_0. Consequently the condition $C\epsilon e^{z_1} \ll 1$ must be checked in any calculation. The lee waves of course will not satisfy a no-slip condition at the ground.

3. Isothermal atmosphere

3.1. *Summation of the isolated waves*

A similarity solution for an isolated wave in a viscous heat-conducting atmosphere was derived by Stevenson *et al.* (1974). The solution has the same form as the

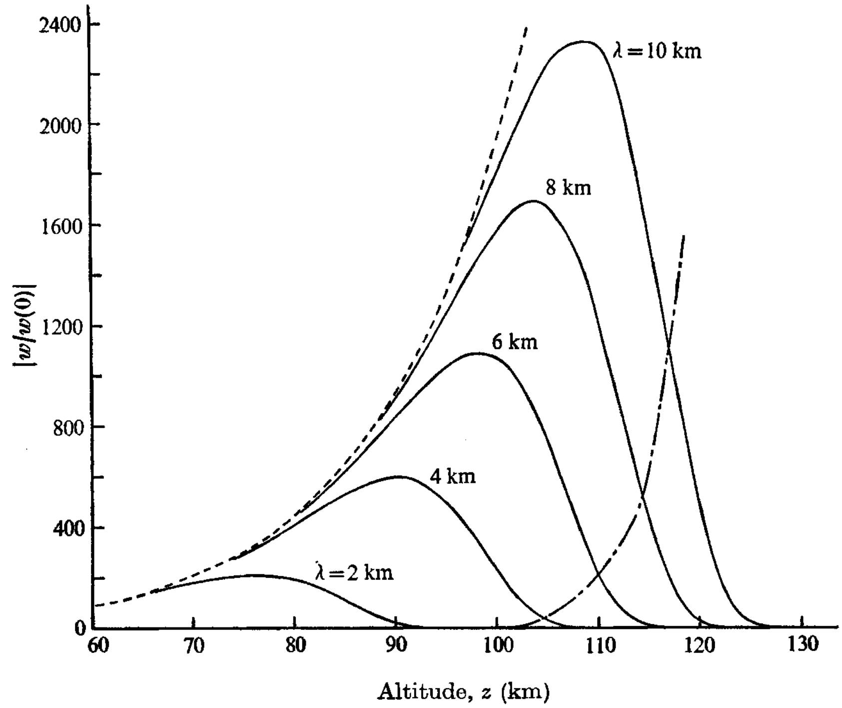

FIGURE 5. Results for larger wavelengths in an isothermal atmosphere having the same background conditions as that in figure 4. The perturbation parameter $\epsilon e^{z_1} > 0{\cdot}1$ to the right of the chain-dashed line.

incompressible solution and the velocity along the wave is again given by (3)–(5). However the r_1 and X_1 are now defined by

$$r_1 = \exp\left[-\gamma X'/(\gamma-1)\right] \tag{27}$$

and

$$X_1 = \left(1+\frac{1}{\sigma^*}\right)\left(\frac{\gamma-1}{\gamma}\right)\left\{\exp\left(\frac{\gamma X'}{\gamma-1}\right)-1\right\}, \tag{28}$$

where σ^* is the Prandtl number and γ is the ratio of the specific heats. Both σ^* and γ are assumed constant. The buoyancy frequency is given by $\omega_0 = (\gamma-1)^{\frac{1}{2}}\, g/a_0$, where a_0 is the sound speed, which is constant. The individual waves are again summed numerically and the amplitude variation with height is shown in figure 4 for several wavelengths. The Boussinesq approximation is reasonable when the wavelength is less than 20 m.

An approximate integration of the isolated waves again produces (12) and, after substituting for r_1 and X_1, the velocity distribution becomes

$$U = U(0)\exp\left[ikx+\left(\frac{\gamma g}{2a_0^2}-ik\cot\theta\right)z-\frac{\nu k^3 a_0^2(1+1/\sigma^*)}{2\omega_0\gamma g\sin^4\theta\cos\theta}\left\{\exp\left(\frac{\gamma g z}{a_0^2}\right)-1\right\}\right] \tag{29}$$

and the temperature distribution takes a similar form. The amplitude–altitude curves from (29) are indistinguishable from the numerical summations shown in figure 4. Results for larger wavelengths are shown in figure 5.

3.2. *A small perturbation analysis*

The linearized continuity and momentum equations for an isothermal atmosphere are

$$-u_0\frac{\partial\rho}{\partial x}+w\frac{\partial\rho_0}{\partial z}+\rho_0\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)=0, \tag{30}$$

$$\rho_0u_0\frac{\partial u}{\partial x}-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial z^2}+(2\mu+\lambda_s)\frac{\partial^2 u}{\partial x^2}+(\mu+\lambda_s)\frac{\partial^2 w}{\partial x\,\partial z}=0, \tag{31}$$

and
$$\rho_0u_0\frac{\partial w}{\partial x}-\frac{\partial p}{\partial z}-\rho g+\mu\frac{\partial^2 w}{\partial x^2}+(2\mu+\lambda_s)\frac{\partial^2 w}{\partial z^2}+(\mu+\lambda_s)\frac{\partial^2 u}{\partial x\,\partial z}=0, \tag{32}$$

where λ_s is the second coefficient of viscosity and both μ and λ_s are taken as constants. The energy equation and the equation of state are

$$\rho_0u_0c_v\frac{\partial T}{\partial x}-p_0\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+k_s\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}\right)=0 \tag{33}$$

and
$$p-\frac{p_0\rho}{\rho_0}=\frac{a_0^2}{\gamma}\rho_0\frac{T}{T_0}, \tag{34}$$

where T is the temperature perturbation, c_v is the specific heat at constant volume, k_s is the thermal conductivity and the subscript zero refers to the undisturbed background conditions. Again we look for a wave solution of the form

$$w=w(z)\exp ikx,\quad \rho=\rho(z)\exp ikx,\ \text{etc.}$$

ρ is eliminated from the equations using the equation of state, u is eliminated using (30) and then p is eliminated using the energy equation. The momentum equations are now rather long ordinary differential equations in w and T. In view of the results of the individual-wave summation, and from the incompressible solution, the following expressions are chosen for w and T:

$$w(z_1)=w(0)\exp\{Az_1-B[\epsilon(e^{z_1}-1)+C\epsilon^2(e^{2z_1}-1)+\ldots]\} \tag{35}$$

and

$$T(z_1)=S(1+D\epsilon e^{z_1}+E\epsilon^2e^{2z_1}+\ldots)\exp\{Az_1-B[\epsilon(e^{z_1}-1)+\ldots]\}, \tag{36}$$

where S is a constant which is related to the temperature $T(0)$ at $z=0$ by $T(0)=(1+D\epsilon+E\epsilon^2+\ldots)S$. The dimensionless vertical co-ordinate $z_1=\gamma gz/a_0^2$ and $\epsilon=\nu^*k/u_0$. These are substituted into the differential equations and terms of like order in ϵe^{z_1} are equated. It is found that

$$A=\tfrac{1}{2}-i|\{K^2(\omega_0^2/u_0^2k^2+M^2-1)-\tfrac{1}{4}\}^{\frac{1}{2}}|, \tag{37}$$

$$B=\frac{i}{2A\gamma^2}\left\{\frac{(1-\gamma^2A^2M^4)}{1-M^2}\left(\frac{F(1-\gamma)}{\sigma^*M^2}+\frac{\lambda_s}{\mu}+1\right)+F\left(\gamma^2[A^2-K^2(1-M^2)]-1\right)\right\}, \tag{38}$$

$$S=\frac{-ia_0(1-\gamma AM^2)}{\gamma Kc_pM(1-M^2)}w(0) \tag{39}$$

and

$$D=\frac{\gamma M^2}{(A\gamma+1)(1-\gamma AM^2)}\{B(1-\gamma A)-i\gamma K^2F(1-M^2)\}+\frac{i\gamma}{(A\gamma+1)}\left\{AM^2\left(1+\frac{\lambda_s}{\mu}\right)-\frac{F(A+1)}{\sigma^*}\right\}, \tag{40}$$

where $K=ka_0^2/\gamma g$, $M=u_0/a_0$ and $F=1-(A^2/K^2)$.

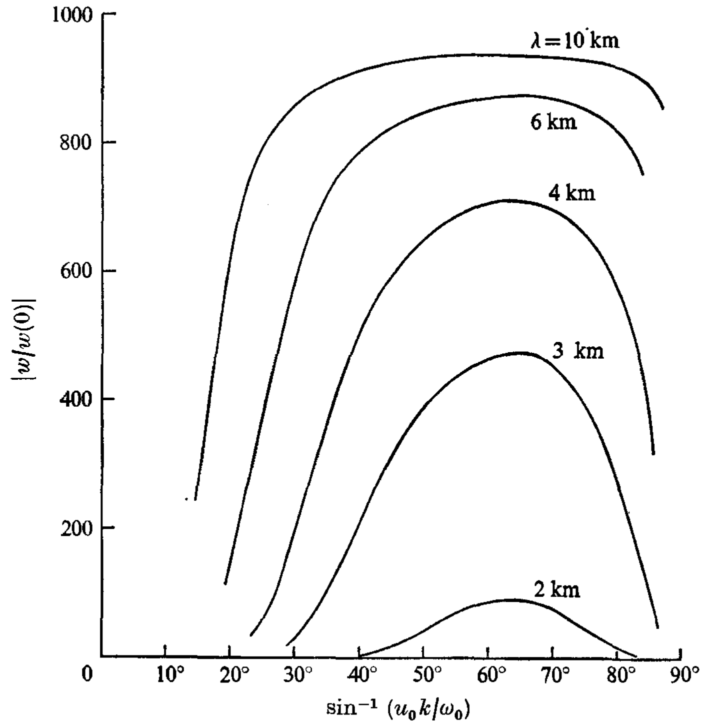

FIGURE 6. The variation in the perturbation $|w(z)|$ with the frequency ratio $u_0 k/\omega_0$ for several wavelengths, showing the ratio of the perturbation at $z = 90$ km to that at $z = 0$. The conditions are the same as those in figure 4.

If A and B are expanded in powers of K^{-1} then

$$A = \tfrac{1}{2} - iK\cot\theta\{1 + O(K^{-1})\} \tag{41}$$

and

$$B = \frac{K}{2\sin^3\theta\cos\theta}\left(1+\frac{1}{\sigma^*}\right)\{1+O(K^{-1})\}. \tag{42}$$

These leading terms give the same equation as that from the superposition of the individual waves, equation (29). The region in which the perturbation parameter $\epsilon e^{z_1} > 0{\cdot}1$ is to the right of the chain-dashed line in figure 5. The way in which the angle θ or the frequency ratio $u_0 k/\omega_0$ affects the wave amplitude is given by (29) and specific examples are shown in figure 6.

The altitude z_m at which the velocity perturbation $|w(z)|$ reaches a maximum, from (29), is

$$z_m = \frac{a_0^2}{\gamma g}\log_e\left\{\frac{\gamma g\omega_0\sin^4\theta\cos\theta}{a_0^2\nu^* k^3(1+\sigma^{*-1})}\right\}. \tag{43}$$

This equation has been used to evaluate the curves in figures 7 and 8.

Pitteway & Hines (1963) studied the viscous problem as a perturbation of the inviscid solution. Their perturbation equations have the solution

$$w(z) = w_I(1 - \nu^* k^3 a_0^2 e^{z_1}/2\gamma g\omega_0\sin^4\theta\cos\theta), \tag{44}$$

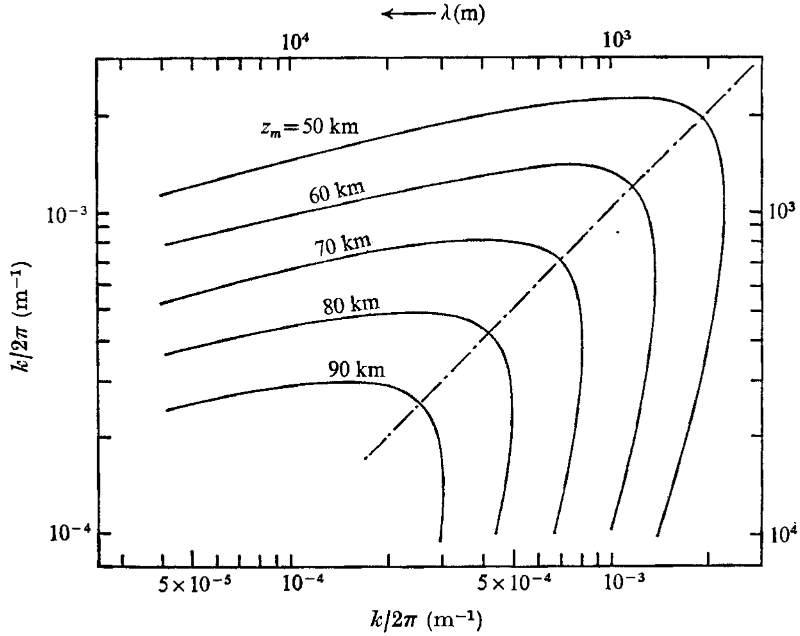

FIGURE 7. The altitude at which the perturbations are a maximum. $z = 0$ corresponds to 14 km in the ISA. λ_z and k_z are the z-direction wavelength and wavenumber respectively. ——, equation (43); —·——·—, $\lambda = \lambda_z$.

where w_I is the inviscid solution without reflected waves. From this equation the altitude at which the velocity is a maximum is given by

$$z_m = \frac{a_0^2}{\gamma g} \log_e \left\{ \frac{2}{3} \frac{\gamma g \omega_0 \sin^4 \theta \cos \theta}{a_0^2 \nu^* k^3} \right\}. \tag{45}$$

Equations (45) and (43) differ by a factor of $\frac{2}{3}$ and a conduction term $(1 + \sigma^{*-1})$. The erroneous $\frac{2}{3}$ in (45) arises because the correction to the inviscid solution is no longer small at the maximum amplitude position.

The analysis presented in the present paper is not a perturbation of the inviscid solution and is more general than that of Pitteway & Hines. The present solution reduces to (44) only if the following extra simplifications are imposed: (*a*) $K^{-1} \ll 1$, (*b*) $B\epsilon e^{z_1} \ll 1$, which is equivalent to assuming that the solution is a perturbation of the inviscid solution, and (*c*) thermal conduction effects are neglected.

Yeh & Liu (1974) have found z_m by considering the energy dissipated within the atmosphere. At a particular position in the atmosphere the change in amplitude of a wave is written as $\exp(Az_1 - \xi z_1)$, where $\exp Az_1$ is the exponential increase in the inviscid solution. ξ is a damping coefficient given by the time-averaged energy dissipation divided by twice the vertical component of the energy flux for an inviscid atmosphere. The altitude at which the amplitude is a maximum occurs when $\xi = A$. For large K, Yeh & Liu's solution for the position of maximum amplitude is the same as (43).

Midgley & Liemohn's (1966) numerical solution is for a realistic atmosphere and the viscosity variation is different from that in the present paper. Even so,

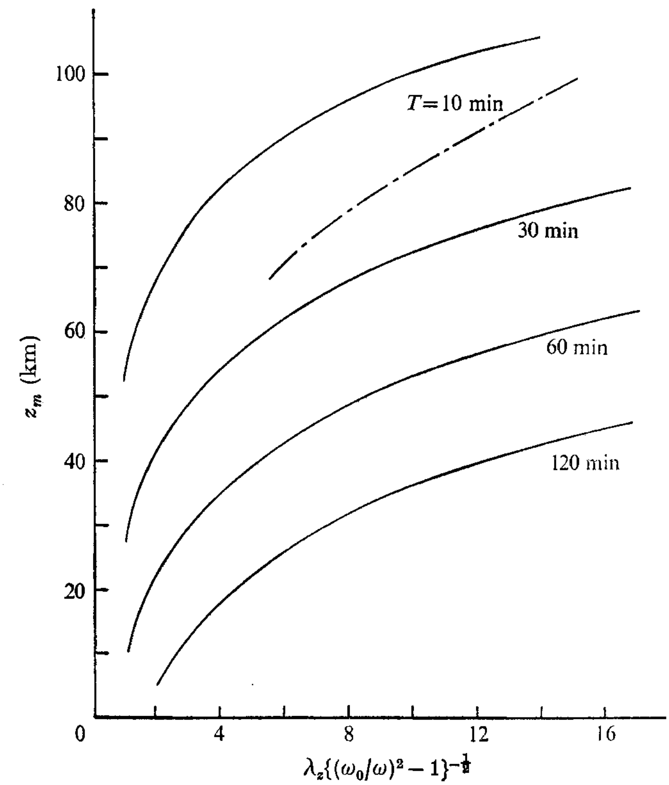

FIGURE 8. The altitude z_m at which the velocity $|w(z)|$ is a maximum, for several periods of oscillation. $T = 2\pi/u_0 k$; λ_z is the wavelength in the z direction. ——, equation (43); —·—·—, Midgeley & Liemohn (1966), $T = 10$ min.

the positions of the maximum amplitudes, shown in figure 8, are not too far from those predicted by (43). Midgley & Liemohn also have the perturbations approaching zero at large z as in this paper. However this is unlike the behaviour in the paper by Lindzen (1970), where the perturbations approach a constant value at large heights. Lyons & Yanowitch (1974) also have perturbations approaching a constant value but their solution is for Prandtl numbers which are an order of magnitude less than that used here. Their isothermal-atmosphere solution is for the disturbance over a flat surface which performs vertical oscillations and k is zero. They do indicate that at larger Prandtl numbers the behaviour should be similar to that of Yanowitch's (1967) incompressible solution, for which the perturbations do approach zero at large heights.

The present theory has not included the effects of the reflected waves considered by Yanowitch (1967) or Warren (1972). However the largest vertical wavelengths used for both the incompressible and the isothermal case are about 10 km. This corresponds to ratios of the amplitudes of the reflected and incident waves of 3×10^{-6} from Yanowitch's incompressible theory and $1{\cdot}4 \times 10^{-2}$ from Warren's isothermal theory. Thus for both cases the reflected waves are of negligible magnitude.

4. Conclusions

It has been shown that the previous isolated-wave solutions of Thomas & Stevenson may be superimposed to calculate a wave system which is sinusoidal in the horizontal direction. Although a constant viscosity was used in the summations for incompressible fluid, the theory could accommodate variations in viscosity. The wave summations suggested new small perturbation expansions for the solution of the linearized equations for both an incompressible fluid and an isothermal atmosphere. The first terms in the expansions have been obtained. The equations have been used to calculate the way in which the amplitude varies with height and with the angle at which the energy is propagating.

Groups of waves may be summed in a similar manner. The edges of the groups will attenuate whereas the waves in the central regions of the groups will initially amplify provided that the wavelength is sufficiently large. Simple examples were given by Stevenson *et al.* (1974).

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